Harvard University
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ES128

Computational Solid and Structural Mechanics

Lecture Notes, Spring 2017

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Contents

Theory

1. Analysis of Discrete Systems

1.1 Trusses in 1D

1.1.1 Introduction

Truss structures, are the simplest possible discrete mechanical systems that we can consider. Trusses consist of a collection of slender elements, which we often call bars. Bars, are elements that we assume to be sufficiently thin, so that they have negligible resistance to torsion, bending or shear and thus we only consider them to resist to axial deformation. In other words, bars are the equivalent of springs, where only axial forces are able to deform the element whereas off-axis forces cause the bar to translate and rotate in space as a rigid body.

A bar's state of deformation is fully determined once the displacements of its endpoints are known. For convenience, we often refer to the endpoints of a bar as "nodes" where the connection between the bar's nodes and the nodes of a finite element is soon to become apparent! In this section, we will show how we can determine the behavior of an entire truss structure by studying the deformation of its building blocks. Before we do so however, let's derive a relationship between the nodal displacements of each bar and the forces acting on them.

1.1.2 The stiffness matrix

Figure 1.1 shows an example of a truss structure, from which we isolate an arbitrary bar and study its deformation. It is useful to note that even though we can decompose the forces and displacements to their components in the $x-y$ coordinate system (as shown in Figure 1.1), it is not necessary. Since a bar can only deform under the influence of axial forces, we can work on a coordinate system $\xi-\eta$ whose ξ axis lies along the axis of the bar. Doing so, any relationships that we derive will be general and will apply to any bar in any structure, independent of its actual orientation in the global system within the truss structure.

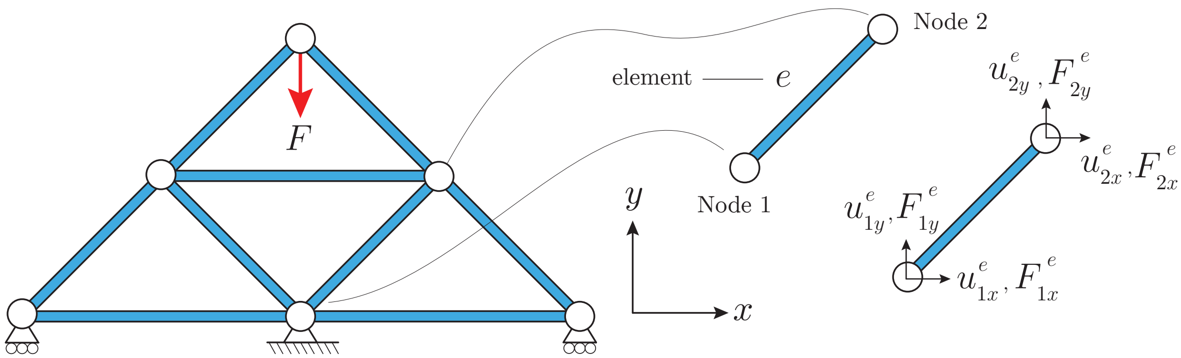

Figure 1.1: A truss structure in which each bar behaves as a finite element

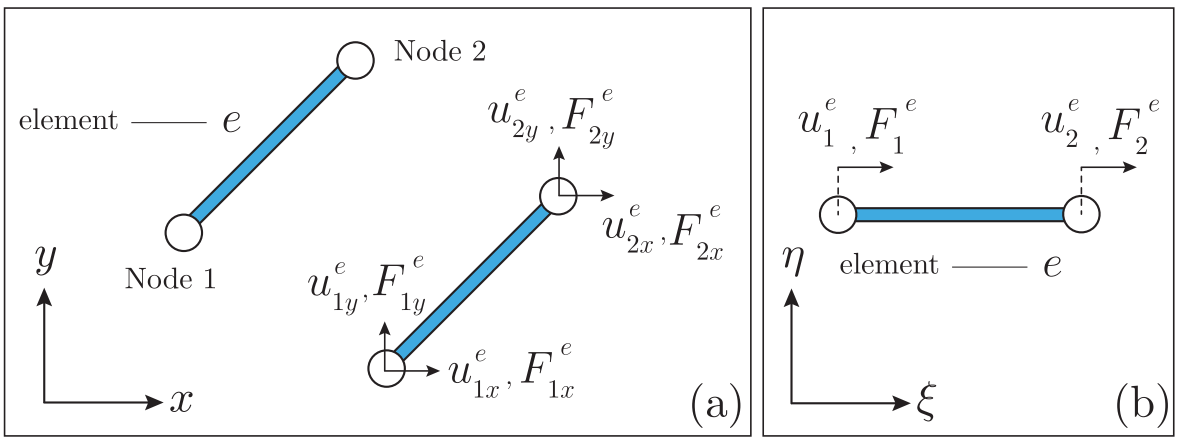

Figure 1.2: The nodal displacements and forces of a bar in the global $x - y$ coordinate system (left) and in the local $\xi - \eta$ coordinate system with the ξ axis aligned with the axis of the bar

The governing equations for the bar are derived as follows. We begin from the **equilibrium equations** which take the simple form,

$$F_1^e + F_2^e = 0 \Rightarrow F_1^e = -F_2^e \tag{1.1}$$

The bar is assumed to be made of a linear elastic material whose **constitutive behavior**[1] is given by Hooke's law as,

$$\sigma^e = \frac{F^e}{A^e} = E^e \cdot \varepsilon^e \Rightarrow F^e = E^e \cdot A^e \cdot \varepsilon^e \tag{1.2}$$

Finally, we have to make sure that when the bar deforms, its deformation is compatible with its nodal displacements in order to maintain continuity and avoid gaps or overlaps. This statement is expressed through the **kinematic** or **compatibility** equations which for this example also have a simple form,

$$\varepsilon^e = \frac{\delta^e}{L^e} = \frac{u_2^e - u_1^e}{L_e} \tag{1.3}$$

[1]The terms constitutive behavior, constitutive law, stress–strain relationship are being used interchangeably throughout the text.

Now, we can just combine equations (1.1–1.3) to derive,

$$F_1^e = -F_2^e = -\underbrace{\frac{E^e \cdot A^e}{L^e}}_{k^e}(u_2^e - u_1^e) \tag{1.4}$$

where the quantity k^e is often referred to as the **stiffness** of the bar, and is the direct equivalent to the stiffness of a spring.

$$\begin{bmatrix} F_1^e \\ F_1^e \end{bmatrix} = k^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} \Rightarrow \boxed{\mathbf{F}^e = \mathbf{K}^e \cdot \mathbf{d}^e} \tag{1.5}$$

where

$$\mathbf{F}^e = \begin{bmatrix} F_1^e \\ F_1^e \end{bmatrix} \quad , \quad \mathbf{K}^e = k^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad , \quad \mathbf{d}^e = \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}$$

The quantity $\mathbf{K}^e$ is referred to as the **Stiffness Matrix** of the element (bar), $\mathbf{F}^e$ represents the vector of the nodal forces whereas $\mathbf{d}^e$ is the vector of the nodal displacements.

> **Remark 1** Note that the stiffness matrix $\mathbf{K}^e$ is **symmetric** since $\mathbf{K}^e = \mathbf{K}^{e\mathrm{T}}$ but also **singular** since $\det(\mathbf{K}^e) = 0$ and therefore it is not invertible.

Discussion 1 As a consequence of the fact that $\mathbf{K}^e$ is singular, we **cannot** solve with respect to the unknown nodal displacements in the traditional way of inverting $\mathbf{K}^e$ so that $\mathbf{d}^e = (\mathbf{K}^e)^{-1} \cdot \mathbf{F}^e$. But why? What does this mean, and what is the physical interpretation of a singular stiffness matrix? The reason why $\mathbf{K}^e$ turned out to be singular is because so far, the bar is completely unconstrained in space which means that it can attain multiple (actually infinite) positions in space for the same choice of nodal forces. Give it a try! Equation (1.4) suggests that both forces depend on the difference between the nodal displacements, suggesting that if $\mathbf{d}^{e*} = [u_1^{e*}, u_2^{e*}]^{\mathrm{T}}$ is a solution, then $\mathbf{d}^{e**} = \mathbf{d}^{e*} + c$ is also a solution $\forall c \in \mathbb{R}$.

Let us now try to generalize the procedure followed in the context of 1 bar, and analyze the behavior of more complicated systems. The steps towards this generalization will be demonstrated through the following example.

1.1.3 From Local to Global: Assembly

■ **Example 1** Consider the system composed of two bars, loaded by external forces as shown in Figure 1.3. Our goal is to relate the displacements u_i to the external forces f_i where $i = 1, 2, 3$.

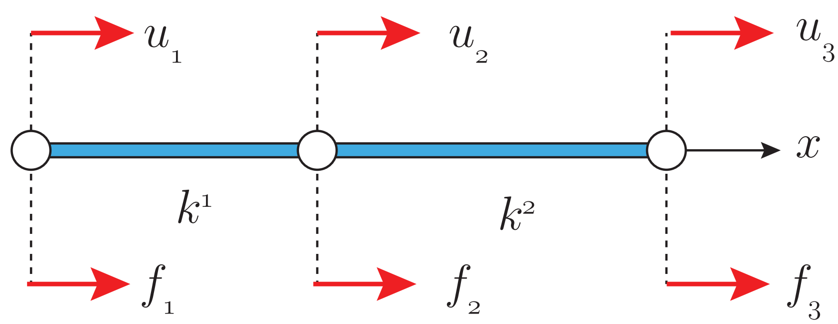

Figure 1.3: A system composed of two bars

(a). Preprocessing: We begin by breaking up the system to its individual components[2] which in this case, are the two bars. Each bar is now a **finite element** and the elements are connected through the **nodes**.

(b). Analysis: Next, we have to develop the governing equations for each one of the individual elements that the system was broken into. In this simple case, the truss system was broken into two individual bars which means that the governing equations for both of the bars are the same. In particular, according to expression (1.5) we can write,

$$\mathbf{F}^e = \mathbf{K}^e \cdot \mathbf{d}^e \quad , \quad e = 1, 2$$

Having written down the equations that describe the behavior of each bar we now

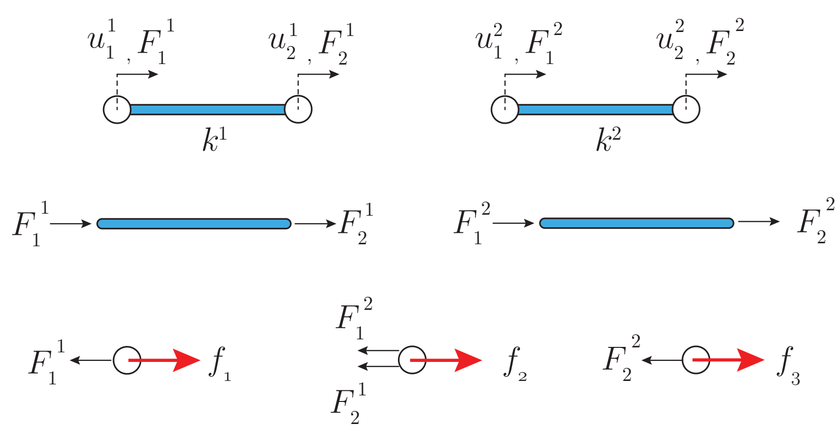

Figure 1.4: Splitting the truss structure into its components. Free body diagrams and force balance

have to *assemble* them in such a way, that the equations would describe the behavior of the whole structure instead of the behavior of the individual building blocks. This can

[2] This process will be extended to continuum systems where it is referred to as **discretization**.

be made possible through a procedure that is commonly referred to as the **Assembly** operation. Let's write down the equations for each bar in matrix form to help us how this operation works. We have,

$$\begin{bmatrix} F_1^1 \\ F_2^1 \end{bmatrix} = k^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} \qquad \begin{bmatrix} F_1^2 \\ F_2^2 \end{bmatrix} = k^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} \tag{1.6}$$

Now comparing Figures 1.3-1.4 we can conclude that the "missing link" between the two systems of equations is essentially,

$$\begin{aligned} &u_1^1 = u_1 \\ &u_2^1 = u_1^2 = u_2 \\ &u_2^2 = u_3 \end{aligned} \tag{1.7}$$

where displacements without superscripts refer to the structure's displacements as shown in Figure 1.3. The equations in (1.7) provide a link between the **local** and **global** variables. It is impossible to solve any problem using the Finite Element Method, without this link! In fact, this link is exactly what makes the Finite Element Method work in the first place. However, it is rather uncommon to present this link in the form of equations such as (1.7). Instead, we introduce a *matrix* which we call the **Element Connectivity** matrix/chart which for our system takes the following form,

Table 1.1: The element connectivity matrix for the simple two-bar truss system in Figure 1.3

Element #	Node 1	Node 2
1	1	2
2	2	3

Now back to our problem, in view of equations (1.7), the systems in (1.6) may be written as,

$$\begin{bmatrix} F_1^1 \\ F_2^1 \end{bmatrix} = k^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \begin{bmatrix} F_1^2 \\ F_2^2 \end{bmatrix} = k^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \tag{1.8}$$

Now notice that by expanding the matrices in (1.8) we can write,

$$\underbrace{\begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{bmatrix}}_{\mathbf{F}^1_{\text{exp}}} = k^1 \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{K}^1_{\text{exp}}} \cdot \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{d}} \quad , \quad \underbrace{\begin{bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{bmatrix}}_{\mathbf{F}^2_{\text{exp}}} = k^2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{\mathbf{K}^2_{\text{exp}}} \cdot \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{d}} \tag{1.9}$$

where now the vector $[u_1 \ u_2 \ u_3]^{\mathrm{T}}$ is the **global displacement vector**. To complete the assembly process we have to invoke force balance as shown in Figure 1.4 in order to relate each element's nodal forces $F_1^1, F_1^2, F_1^2, F_2^2$ with the global internal forces f_1, f_2, f_3. It is straightforward to conclude that,

$$\begin{aligned} f_1 - F_1^1 = 0 &\Rightarrow f_1 = F_1^1 \\ f_2 - F_2^1 - F_1^2 = 0 &\Rightarrow f_2 = F_2^1 + F_1^2 \\ f_3 - F_2^2 = 0 &\Rightarrow f_3 = F_2^2 \end{aligned} \tag{1.10}$$

In other words,

$$\underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{\mathbf{F}} = \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{bmatrix}$$

However, recall that $\mathbf{F}^1_{\mathrm{exp}} = \mathbf{K}^1_{\mathrm{exp}} \cdot \mathbf{d}$ and $\mathbf{F}^2_{\mathrm{exp}} = \mathbf{K}^2_{\mathrm{exp}} \cdot \mathbf{d}$, which means that,

$$\mathbf{F} = \mathbf{K}^1_{\mathrm{exp}} \cdot \mathbf{d} + \mathbf{K}^2_{\mathrm{exp}} \cdot \mathbf{d} = \left(\mathbf{K}^1_{\mathrm{exp}} + \mathbf{K}^2_{\mathrm{exp}}\right) \cdot \mathbf{d} = \mathbf{K} \cdot \mathbf{d}$$

and

$$\mathbf{K} = \sum_e \mathbf{K}^e_{\mathrm{exp}} = \begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \tag{1.11}$$

> **Remark 2** Note that the global stiffness matrix $\mathbf{K}$ is **symmetric** since $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$ but also **singular** since $\det(\mathbf{K}) = 0$ and therefore it is not invertible. The latter, as pointed out before, is due to the fact that no boundary conditions where applied to constrain the truss structure from moving as a rigid body. Lastly, the equations in $\mathbf{F} = \mathbf{K} \cdot \mathbf{d}$ are essentially the three equilibrium equations at the three nodes of the structure.

At this point, the **Assembly** operation has been completed. We have successfully converted the two systems of equations that governed the behavior of each bar, to one (1) system of equations that describes the behavior of the entire structure.

(c). **Solution**: We now proceed to the last and final step of our analysis, solving the equations. We have already pointed out that the global stiffness matrix $\mathbf{K}$ is singular due to the fact that no boundary conditions where applied to constrain the truss structure in space and eliminate rigid body motions. We consider the two following scenarios for boundary conditions.

In both cases, the boundary conditions will be enforced using the **partition approach**. According to this approach, we partition the nodes as follows:

- **E-nodes**: Essential Nodes, or nodes for which the displacements are known
- **F-nodes**: Free Nodes, or nodes for which the displacements are unknown

This means, that every time we apply boundary conditions of any kind in a structure, we will partition the global displacement vector $\mathbf{d}$ as,

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_E \\ \mathbf{d}_F \end{bmatrix}$$

and note that $\mathbf{d}_E$, $\mathbf{d}_F$ are still vectors in the general case.

> **Remark 3** Notice that is impossible to simultaneously prescribe both the displacement and the force for any node, or any point in the structure. Recall from elementary solid mechanics that if we prescribe the displacement of a point within a structure then the (reaction) force at that point will be unknown. Equivalently, if the force is prescribed at any point in the structure, then the displacement of that point will be unknown,

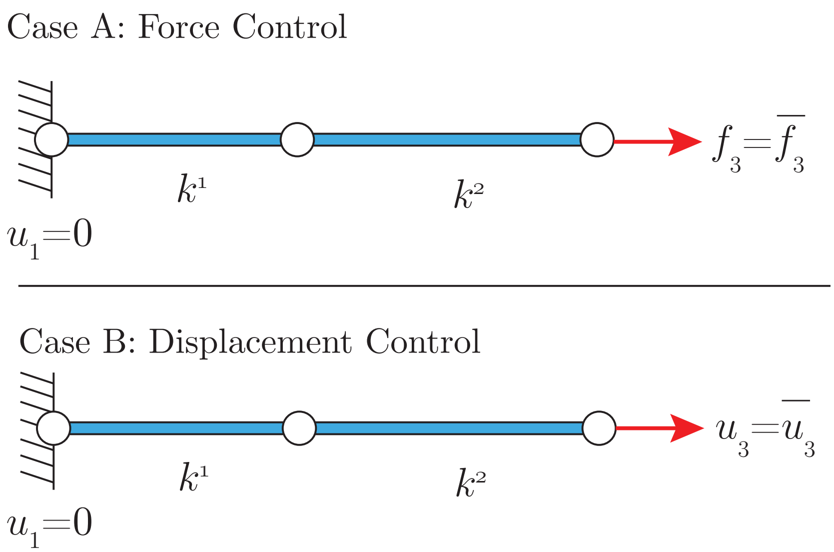

Figure 1.5: Two cases of boundary conditions for the two-bar truss structure.

> prior to the solution of the problem.

The above remark implies that if we partition the global force vector $\mathbf{F}$ in the same way we did for $\mathbf{d}$ and write,

$$\begin{bmatrix} \mathbf{F}_E \\ \mathbf{F}_F \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{EE} & \mathbf{K}_{EF} \\ \mathbf{K}_{FE} & \mathbf{K}_{FF} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{d}_E \\ \mathbf{d}_F \end{bmatrix}$$

then the reaction forces $\mathbf{F}_E$ that correspond to the nodes at which we prescribed displacements will be unknown. Furthermore, the displacements $\mathbf{d}_F$ of the nodes at which we applied forces will also be unknown. Now that we know what is known and what is not, we may proceed by solving the system of equations in the following two steps. First, we solve for $\mathbf{d}_F$ from,

$$\mathbf{F}_F = \mathbf{K}_{FE} \cdot \mathbf{d}_E + \mathbf{K}_{FF} \cdot \mathbf{d}_F \Rightarrow \boxed{\mathbf{d}_F = \mathbf{K}_{FF}^{-1} \cdot (\mathbf{F}_F - \mathbf{K}_{FE} \cdot \mathbf{d}_E)} \tag{1.12}$$

Next, with $\mathbf{d}_E$ known, we can solve for $\mathbf{F}_E$ as,

$$\boxed{\mathbf{F}_E = \mathbf{K}_{EE} \cdot \mathbf{d}_E + \mathbf{K}_{EF} \cdot \mathbf{d}_F} \tag{1.13}$$

Now let us consider the two particular cases shown in Figure 1.5 and solve for $\mathbf{d}_F$ and $\mathbf{F}_E$.

(A). Here,

$$\left[\begin{array}{c|c} \mathbf{K}_{EE} & \mathbf{K}_{EF} \\ \hline \mathbf{K}_{FE} & \mathbf{K}_{FF} \end{array}\right] = \left[\begin{array}{c|cc} k^1 & -k^1 & 0 \\ \hline -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{array}\right], \quad \left[\begin{array}{c} \mathbf{F}_E \\ \hline \mathbf{F}_F \end{array}\right] = \left[\begin{array}{c} f_1 \\ \hline f_2 \\ f_3 \end{array}\right] = \left[\begin{array}{c} f_1 \\ \hline 0 \\ \bar{f}_3 \end{array}\right]$$

and

$$\left[\begin{array}{c} \mathbf{d}_E \\ \hline \mathbf{d}_F \end{array}\right] = \left[\begin{array}{c} u_1 \\ \hline u_2 \\ u_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline u_2 \\ u_3 \end{array}\right]$$

which we can solve using equations (1.12) and (1.13) to find,

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \bar{f}_3 \begin{bmatrix} \frac{1}{k^1} \\ (\frac{1}{k^1} + \frac{1}{k^2}) \end{bmatrix} \quad , \quad f_1 = -\bar{f}_3$$

(B). In this case,

$$\left[\begin{array}{c|c} \mathbf{K}_{EE} & \mathbf{K}_{EF} \\ \hline \mathbf{K}_{FE} & \mathbf{K}_{FF} \end{array}\right] = \left[\begin{array}{cc|c} k^1 & 0 & -k^1 \\ 0 & k^2 & -k^2 \\ \hline -k^1 & -k^2 & k^1 + k^2 \end{array}\right] \quad , \quad \left[\begin{array}{c} \mathbf{F}_E \\ \hline \mathbf{F}_F \end{array}\right] = \left[\begin{array}{c} f_1 \\ f_3 \\ \hline f_2 \end{array}\right] = \left[\begin{array}{c} f_1 \\ f_3 \\ \hline 0 \end{array}\right]$$

and

$$\left[\begin{array}{c} \mathbf{d}_E \\ \hline \mathbf{d}_F \end{array}\right] = \left[\begin{array}{c} u_1 \\ u_3 \\ \hline u_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ \bar{u}_3 \\ \hline u_2 \end{array}\right]$$

and notice that we rearranged the equations in all the vectors and matrices in order to make the grouping consistent. The solution for these boundary conditions is,

$$u_2 = \frac{k^2}{k^1 + k^2} \bar{u}_3 \quad , \quad \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \bar{u}_3 \frac{k^1 \, k^2}{k^1 + k^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

■

Discussion 2 The stiffness matrix $\mathbf{K}$ is extremely important in all aspects of the finite element method. However, even though we can readily interpret the concept of forces and displacements, it is not entirely clear how the spring constant k of a linear spring that obeys Hooke's law generalizes to a matrix $\mathbf{K}$. What is the physical significance of $\mathbf{K}$ and in particular of element K_{ij}?

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1N} \\ K21 & K_{22} & \dots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ KN1 & K_{N2} & \dots & K_{NN} \end{bmatrix}$$

The component K_{ij} represents the force i due to a unit displacement at note j while keeping all other nodes fixed

■ **Example 2** In this example, we are considering a slightly more complicated problem, where a system of 3 trusses is subjected to axial deformation through a force P. The configuration considered in this problem is shown in Figure 1.6 and we are requested to determine the unknown nodal displacements and reaction forces with respect to P.

Following the procedure outlined in example 1, we begin by writing the stiffness matrices for each element in the structure as follows,

$$\mathbf{K}^{(i)} = k^i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad , \quad i = 1, 2, 3 \tag{1.14}$$

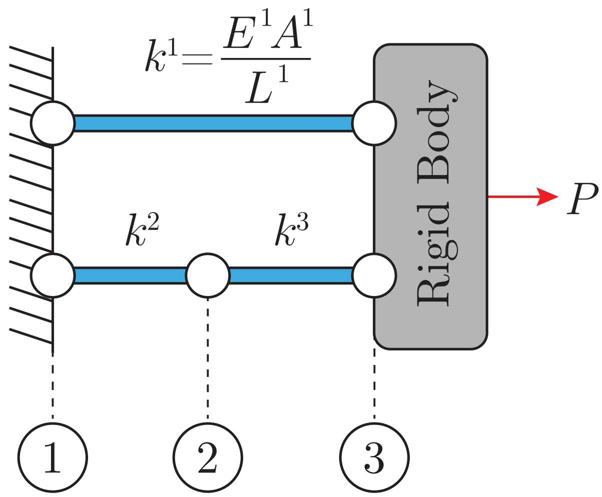

Figure 1.6: An example of 3 bars connected to a rigid wall to the left and to a rigid body to the right, loaded with a point force P in the x-direction

where $k^i = E^i \cdot A^i / L^i$. At this point, we could pretend that since there are 5 nodes in total in the problem, there are also 5 degrees of freedom, assuming that each node can move independently from any other. However, the boundary conditions in both edges constrain the nodes attached to them to move "as one" implying that the actual degrees of freedom for this problem are just 3, as shown in Figure 1.6. We now construct the global stiffness matrix assembling the individual stiffness matrices for each element to arrive at,

$$\mathbf{K} = \sum_e \mathbf{K}^{(i)} = \begin{bmatrix} k^1 + k^2 & -k^2 & -k^1 \\ -k^2 & k^2 + k^3 & -k^3 \\ -k^1 & -k^3 & k^1 + k^3 \end{bmatrix} \tag{1.15}$$

The nodal displacements and reaction forces are simply,

$$\mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad , \quad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{1.16}$$

The boundary conditions, can now be expressed in terms of the u_i and F_i as,

$$u_1 = 0 \quad , \quad F_2 = 0 \quad , \quad F_3 = P$$

Therefore, we have,

$$\mathbf{d}_E = \begin{bmatrix} 0 \end{bmatrix} \quad , \quad \mathbf{F}_E = \begin{bmatrix} F_1 \end{bmatrix} \quad , \quad \mathbf{d}_F = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \quad , \quad \mathbf{F}_F = \begin{bmatrix} 0 \\ P \end{bmatrix} \tag{1.17}$$

and,

$$\mathbf{K} = \left[\begin{array}{c|cc} k^1 + k^2 & -k^2 & -k^1 \\ \hline -k^2 & k^2 + k^3 & -k^3 \\ -k^1 & -k^3 & k^1 + k^3 \end{array}\right] \tag{1.18}$$

According to example 1 the first step is to solve with respect to $\mathbf{d}_F$ using equation (1.12). We have,

$$\mathbf{d}_F = \mathbf{K}_{FF}^{-1} \cdot (\mathbf{F}_F - \mathbf{K}_{FE} \cdot \mathbf{d}_E) \Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{P}{k^1 k^2 + k^2 k^3 + k^1 k^3} \begin{bmatrix} k^3 \\ k^2 + k^3 \end{bmatrix}$$

and last, we solve for $\mathbf{F}_E$ using (1.13) to find,

$$\mathbf{F}_E = \mathbf{K}_{EE} \cdot \mathbf{d}_E + \mathbf{K}_{EF} \cdot \mathbf{d}_F =$$

$$(k^1 + k^2) \cdot 0 + \frac{P}{k^1 k^2 + k^2 k^3 + k^1 k^3} \begin{bmatrix} -k^2 & -h^1 \end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} k^3 \\ k^2 + k^3 \end{bmatrix} \Rightarrow$$

$$F_1 = \frac{k^2 k^3 + k^1 k^2 + k^1 k^3}{k^1 k^2 + k^2 k^3 + k^1 k^3} P = -P$$

■

1.2 Trusses in 2D

In this section, we will generalize the procedure used to solve truss problems in 1D, extending it to apply for truss members randomly oriented in the 2D space. In the previous section, when we first introduced the methodology to describe the deformation of a single truss member, we briefly mentioned that the orientation of the bar does not matter in the analysis, as long as we do all of our calculations in a local coordinate system whose one axis is aligned with the axis of the bar. While this may be true in the case of 1 bar (*see* Figure 1.2), things are a bit more complicated when we want to determine the behavior of a whole truss structure in 2D. We still work on a local coordinate system in the context of individual bars but in order to assemble the global stiffness matrix we have to "translate" our expressions back to the global coordinate system and thus account for the bar's orientation. Before we discuss the systematic methodology to cope with 2D trusses let us outline the underlying assumptions for every example or problem associated with trusses:

- The truss members (bars) are connected only at their ends
- The connections are frictionless pins and thus do not develop moments upon loading
- The structure can only "accept" external loads at the pins. All loads drawn as if they were distributed over a finite region of the structure are also assumed to be exerted at the closest relevant pins
- The weight of each truss member is neglected

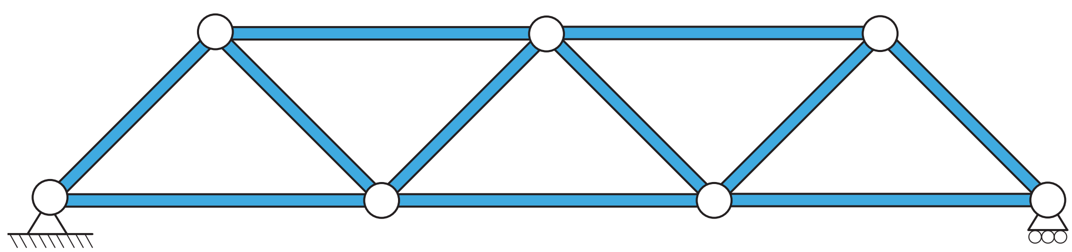

Figure 1.7: A typical truss structure than can be loaded in both x and y directions

1.2.1 The stiffness matrix of a randomly oriented bar

Recall that each truss member can only be subjected to tension or compression along its axis. However, the fact that the orientation of each truss member is now allowed to be arbitrary implies that upon loading, the nodal displacements along ξ in the local coordinate system, would in general result in displacements in both x and y in the global coordinate system. In an effort to be systematic in our notation let us always denote as x, y the local coordinate system and as ξ, η the local coordinate system as shown in Figure 1.2. A more detailed figure regarding the displacements of a truss member due to deformation, in the local and global coordinate systems is shown in Figure 1.8 that follows.

For a bar oriented at an angle θ with respect to the x-axis of the global system[3], it is straightforward to write,

$$u^e_{1\xi} = u^e_{1x} \cos\theta + u^e_{1y} \sin\theta \tag{1.19}$$

$$u^e_{2\xi} = u^e_{2x} \cos\theta + u^e_{2y} \sin\theta \tag{1.20}$$

[3]Note: The local coordinate system always has the axis ξ aligned with the axis of the bar

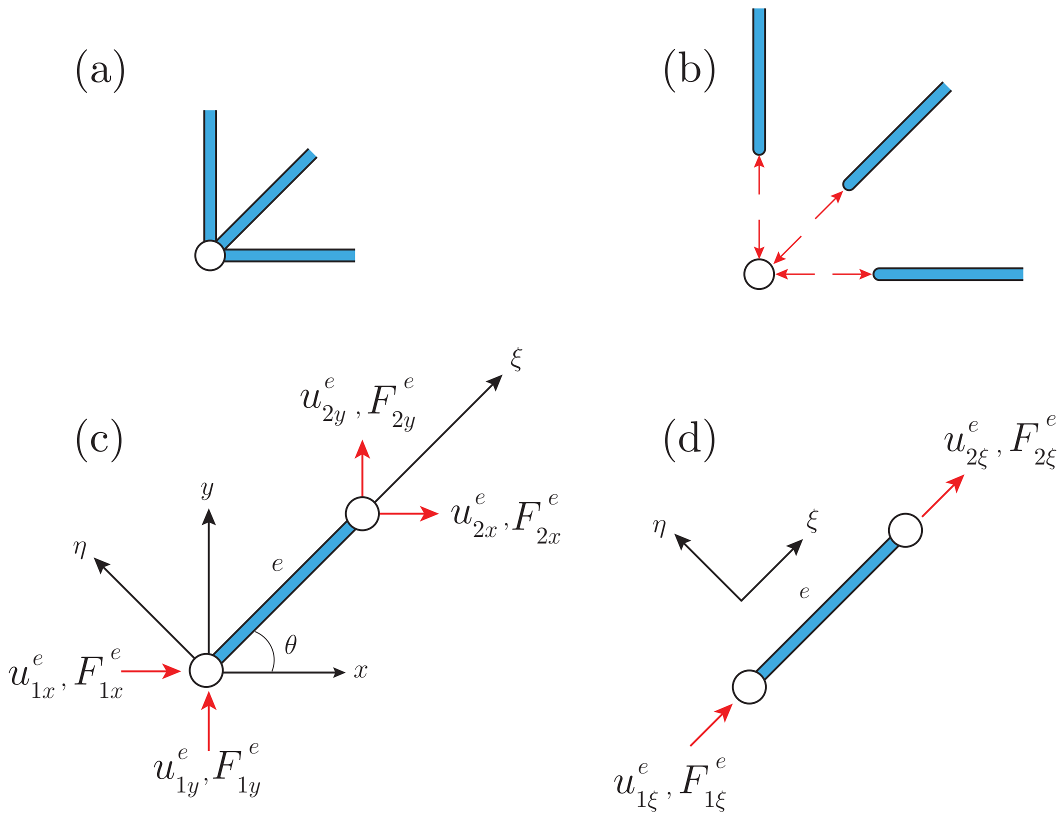

Figure 1.8: (a) and (b): A schematic representation of the internal forces in the nodes of a bar. (c) and (d): The nodal displacements of a tilted bar in x and y directions with respect to the global coordinate system can always be calculated from the corresponding displacements in the local coordinate system ξ,η.

Notice however, that the above equations can be expressed in matrix form as

$$\mathbf{d}^e_{\xi,\eta} = \mathbf{T}^e(\theta) \cdot \mathbf{d}^e_{x,y} \Rightarrow \begin{bmatrix} u^e_{1\xi} \\ u^e_{2\xi} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} u^e_{1x} \\ u^e_{1y} \\ u^e_{2x} \\ u^e_{2y} \end{bmatrix} \tag{1.21}$$

where matrix $\mathbf{T}$ is a rotation matrix, associated with the truss member under consideration and depends only on the orientation angle θ. In the equation that follows, the local c.s. $\xi - \eta$ will be denoted as $\mathcal{L}$ and the global c.s. $x - y$ as $\mathcal{G}$ for convenience. Now recall,

$$\mathbf{F}^e_{\mathcal{L}} = \mathbf{K}^e_{\mathcal{L}} \cdot \mathbf{d}^e_{\mathcal{L}} \Rightarrow \mathbf{F}^e_{\mathcal{G}} \cdot \mathbf{T}^e(\theta) = \mathbf{K}^e_{\mathcal{L}} \cdot \mathbf{T}^e(\theta) \cdot \mathbf{d}^e_{\mathcal{G}} \Rightarrow \mathbf{F}^e_{\mathcal{G}} = \left[\mathbf{T}^{e\mathrm{T}}(\theta) \cdot \mathbf{K}^e_{\mathcal{L}} \cdot \mathbf{T}^e(\theta)\right] \cdot \mathbf{d}^e_{\mathcal{G}}$$

Now dropping the subscript $\mathcal{G}$, knowing that we refer to the global system we write,

$$\boxed{\mathbf{F}^e = \mathbf{K}^e \cdot \mathbf{d}^e} \tag{1.22}$$

with

$$\boxed{\mathbf{K}^e = \mathbf{T}^{e\mathrm{T}}(\theta) \cdot \mathbf{K}^e_{\mathcal{L}} \cdot \mathbf{T}^e(\theta)} = k^e \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \tag{1.23}$$

or

$$\mathbf{K}^e = k^e \begin{bmatrix} \cos^2\theta & \cos\theta\,\sin\theta & -\cos^2\theta & -\cos\theta\,\sin\theta \\ \cos\theta\,\sin\theta & \sin^2\theta & -\cos\theta\,\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\,\sin\theta & \cos^2\theta & \cos\theta\,\sin\theta \\ -\cos\theta\,\sin\theta & -\sin^2\theta & \cos\theta\,\sin\theta & \sin^2\theta \end{bmatrix}$$

Finally, notice that since $\mathbf{T}^e(\theta)$ is just a rotation matrix it satisfies $\mathbf{T}^e(\theta) \cdot \mathbf{T}^{e\mathrm{T}}(\theta) = \mathbf{I}$ since,

$$\mathbf{T}^e(\theta) \cdot \mathbf{T}^{e\mathrm{T}}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} = (\cos^2\theta + \sin^2\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Now that we were able to form the elemental stiffness matrix for a bar that is randomly oriented in the 2D space, it is just a matter of extending all the elemental stiffness matrices in the structure to perform the assembly operation as described in example 1. The partition method to enforce the boundary conditions applies here as well and the unknown nodal displacements and reaction forces will be determined from (1.12) and (1.13) respectively.

Remark 4 Notice that the elemental stiffness matrix for a randomly oriented bar, apart from the larger matrix dimensions, also has a different form compared to 1D. A convenient way to explain which type of information is stored in each cell of the elemental stiffness matrix is the following,

$$\mathbf{K}^e = \begin{array}{c} \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{matrix} (1x) \\ (1y) \\ (2x) \\ (2y) \end{matrix} \\ \begin{matrix} (1x) & (1y) & (2x) & (2y) \end{matrix} \end{array}$$

It will be useful to keep this in mind when you are dealing with truss problems in 2D. Sometimes, assembling matrices of this form for truss systems with multiple members makes it confusing to think about which component refers to which coordinate.

In what follows, we will solve a relatively simple truss problem to apply the above generalizations.

■ **Example 3** Consider the simple truss structure consisting of two bars, that have their left endpoints pinned on a vertical wall as shown in Figure 1.9. We are requested to determine the unknown displacements u_{2x}, u_{2y}

We begin by writing down the element connectivity matrix which in this case has the form, Now given that $\cos(45°) = -\cos(135°) = \sqrt{2}/2$ we can write down the element stiffness

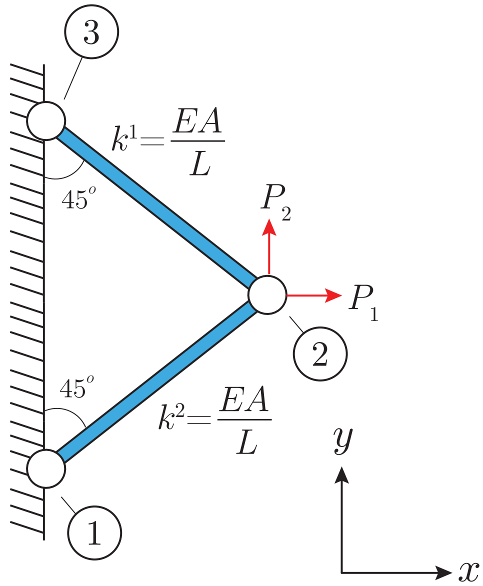

Figure 1.9: An example of 2 bars connected to a rigid wall to the left, loaded with point forces P_1, P_2 in the x and y directions

Element	Node 1	Node 2	Angle θ
1	1	2	45°
2	2	3	135°

Table 1.2: The element connectivity matrix for the truss system in example3

matrix in the global coordinates as,

$$\mathbf{K}^1 = \frac{k}{2}\begin{array}{cc} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} & \begin{array}{c} (1x) \\ (1y) \\ (2x) \\ (2y) \end{array} \\ \begin{array}{cccc} (1x) & (1y) & (2x) & (2y) \end{array} & \end{array} \quad \text{and} \quad \mathbf{K}^2 = \frac{k}{2}\begin{array}{cc} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} & \begin{array}{c} (2x) \\ (2y) \\ (3x) \\ (3y) \end{array} \\ \begin{array}{cccc} (2x) & (2y) & (3x) & (3y) \end{array} & \end{array}$$

Now using the connectivity table along with the elemental stiffness matrices, we may assembly the full stiffness matrix of the structure as,

$$\mathbf{K} = \frac{k}{2}\begin{array}{cc} \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 1+1 & 1-1 & -1 & 1 \\ -1 & -1 & 1-1 & 1+1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} & \begin{array}{c} (1x) \\ (1y) \\ (2x) \\ (2y) \\ (3x) \\ (3y) \end{array} \\ \begin{array}{cccccc} (1x) & (1y) & (2x) & (2y) & (3x) & (3y) \end{array} & \end{array} \Rightarrow \mathbf{K} = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

We may now enforce the boundary conditions and proceed with solving the system of equations. The set of free nodes is only node 2, whereas nodes 1 and 3 are the essential nodes since their displacements are known. Therefore,

$$\mathbf{d}_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{F}_E = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{3x} \\ F_{3y} \end{bmatrix} \quad , \quad \mathbf{d}_F = \begin{bmatrix} u_{2x} \\ u_{2y} \end{bmatrix} \quad , \quad \mathbf{F}_F = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

Hence, we may also rearrange the columns and rows in the stiffness matrix in order to construct the submatrices $\mathbf{K}_{EE}, \mathbf{K}_{EF}, \mathbf{K}_{FE}, \mathbf{K}_{FF}$ as follows,

$$\mathbf{K} = \frac{k}{2} \begin{array}{c} \left[\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ \hline -1 & 1 & -1 & 1 & 2 & 0 \\ -1 & -1 & 1 & -1 & 0 & 2 \end{array}\right] \begin{array}{c} (1x) \\ (1y) \\ (3x) \\ (3y) \\ (2x) \\ (2y) \end{array} \\ \begin{array}{cccccc} (1x) & (1y) & (3x) & (3y) & (2x) & (2y) \end{array} \end{array}$$

and,

$$\mathbf{K}_{EE} = \frac{k}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad , \quad \mathbf{K}_{FF} = \frac{k}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{K}_{EF} = \frac{k}{2} \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \quad , \quad \mathbf{K}_{FE} = \frac{k}{2} \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{bmatrix}$$

We can now solve for $\mathbf{d}_F$ as,

$$\mathbf{d}_F = \mathbf{K}_{FF}^{-1} \cdot (\mathbf{F}_F - \mathbf{K}_{FE} \cdot \mathbf{d}_E) \Rightarrow \mathbf{d}_F = \frac{1}{k} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

Finally, with $\mathbf{d}_F$ known we can now determine the unknown reaction forces at nodes 1 and 3 as,

$$\mathbf{F}_E = \mathbf{K}_{EE} \cdot \mathbf{d}_E + \mathbf{K}_{EF} \cdot \mathbf{d}_F = \frac{k}{2} \frac{1}{k} \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -P_1 & -P_2 \\ -P_1 & -P_2 \\ -P_1 & P_2 \\ P_1 & -P_2 \end{bmatrix}$$

■